<u>Zorn's lemma</u> Let $(\mathbb{P}, \leq)$ be a partially ordered set in which every wellordered subset $X \subseteq \mathbb{P}$ has an upper bound, i.e., there is $a \in \mathbb{P}$ such that $x \leq a$ for all $x \in X$.

Then there is a maximal element in $\mathbb{P}$.

<u>Proof</u> Suppose not. Then, if $X \subseteq \mathbb{P}$ is wellordered and $a \in \mathbb{P}$ is an upper bound for X, there is some $b \in \mathbb{P}$ s.t. $a < b$. Hence, $x < b$ for all $x \in X$.

By the axiom of choice, there is a function $c : \mathcal{P}(\mathbb{P}) \setminus \{\emptyset\} \longrightarrow \mathbb{P}$ that to every non-empty subset $Y \subseteq \mathbb{P}$ picks an element $c(Y) \in Y$.

Let f be the function that to every well-ordered subset $X \subseteq \mathbb{P}$ associates

$$f(X) := c\left(\{b \in \mathbb{P} \mid x < b \text{ for all } x \in X\}\right).$$

So $f(X)$ is a strict upper bound for X.

A set $X \subseteq \mathbb{P}$ is *good* if

(i) X is well-ordered, and

(ii) for all $x_0 \in X$, $x_0 = f(\{y \in X \mid y < x_0\})$.

Also, if $X \subseteq Y \subseteq \mathbb{P}$ are subsets, we say that X is *downwards closed* in Y if for $x \in X$, $y \in Y$, if $y \leq x$ then $y \in X$.

Claim If $X, Y \subseteq \mathbb{P}$ are good, then either X is a downwards closed subset of Y or Y is a downwards closed subset of X.

To see this, let W be the union of all subsets of $X \cap Y$ that are downwards closed in both X and in Y. Then W is clearly the largest subset of $X \cap Y$ that is downwards closed in both X and Y. Also, being a subset of a wellordered set, W is itself well-ordered.

But if $W \subsetneq X$ and $W \subsetneq Y$, we have $f(W) \in X$ and $f(W) \in Y$, whereby $W \cup \{f(W)\} \subseteq X \cap Y$ is a strictly larger set, downwards closed in both X and in Y. This contradicts the maximality of W, and hence either $X = W \subseteq Y$ and X is downwards closed in Y or $Y = W \subseteq X$ and Y is downwards closed in X.

By the claim it follows that if V is the union of all good subsets of $\mathbb{P}$, then V is good and hence the largest good set, which contradicts that also $V \cup \{f(V)\}$ is good. This contradiction proves the theorem. $\square$